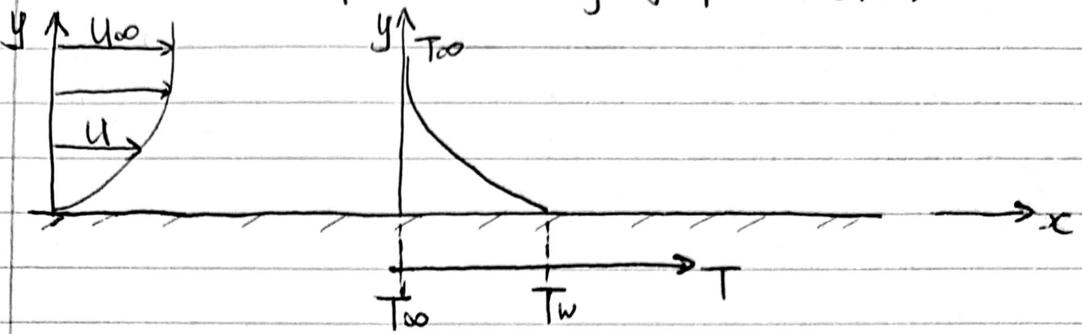
04/02/20

Recap:

When fluid flows over a solid surface, a temperature distribution forms, bridging fluid (T_∞) and the wall (T_w)



The heat exchange between the wall and the fluid can be expressed as the heat conducted away from the wall by the fluid (Fourier's Law)

$$q = -k_f \frac{\partial T}{\partial y}\bigg|_{y=0,\ \text{at the wall}} \qquad k_f: \text{thermal conductivity of the fluid}$$

This defines the convective heat transfer coefficient h

$$-k_f \frac{\partial T}{\partial y}\bigg|_{y=0} = h \cdot (T_w - T_\infty)$$

It can be seen that in order to obtain h, we must first obtain $T(y)$, which gives us $\frac{\partial T}{\partial y}\Big|_{y=0}$

Aim of Lecture 2: how to solve for T distribution.

Lecture 2.

Conservation equations needed to solve for temperature profiles

<1> Conservation of mass — The continuity equation
<2> Conservation of momentum — Navier–Stokes equation
<3> Conservation of energy — The energy equation

<1> and <2> we have learnt in Fluid Mechanics.

Brief review:

<1> Continuity equation (for an incompressible fluid)

$$\boxed{\nabla \cdot \vec{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0}$$

u, v, and w are the x, y, and z components of the velocity vector $\vec{u}$.
In 2D, the equation reduces to $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

<2> x-direction momentum equation (2D, boundary layer)

$$\underbrace{\cancel{\frac{\partial u}{\partial t}}}_{\text{steady state}} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dP}{dx} + \frac{\mu}{\rho}\left(\underbrace{\cancel{\frac{\partial^2 u}{\partial x^2}}}_{\text{small}} + \frac{\partial^2 u}{\partial y^2}\right) + \underbrace{\cancel{g}}_{\text{neglect, not in } x\text{-direction}}$$

reduces to
$$\boxed{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dP}{dx} + \frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2}}$$

we have applied boundary layer approximations

- $\left|\frac{\partial u}{\partial x}\right| \ll \left|\frac{\partial u}{\partial y}\right|$
- $v \ll u$
- $p \neq fn(y)$

Bernoulli equation for the mainstream flow above B.L.

$$\frac{p}{\rho} + \frac{u_\infty^2}{2} = \text{constant.}$$

<3> The energy equation.

→ extend heat conduction equation to allow for fluid motion

$$\underbrace{\rho c_p \frac{\partial T}{\partial t}}_{\substack{\text{energy}\\\text{storage}}} = \underbrace{k \nabla^2 T}_{\substack{\text{heat}\\\text{conduction}}} + \underbrace{\dot{q}}_{\substack{\text{heat}\\\text{generation}}}$$

in 1D: $\rho c_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} + \dot{q}$

with fluid motion, which carries energy, we need an enthalpy term:

$$\boxed{\rho c_p \left(\underbrace{\frac{\partial T}{\partial t}}_{\substack{\text{energy}\\\text{storage}}} + \underbrace{\vec{u}\cdot\nabla T}_{\substack{\text{enthalpy}\\\text{convection}}}\right) = \underbrace{k \nabla^2 T}_{\substack{\text{heat}\\\text{conduction}}} + \underbrace{\dot{q}}_{\substack{\text{heat}\\\text{generation}}}}$$

Note: $\frac{\partial T}{\partial t} + \vec{u}\cdot\nabla T = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} = \frac{DT}{Dt}$

$\frac{D}{Dt}$ is the material derivative.

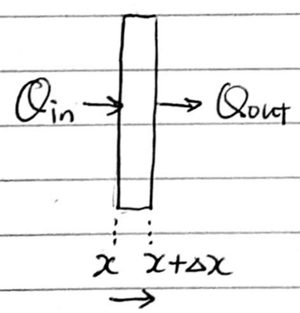
$\frac{DT}{Dt}$ represents change of T of a fluid particle as it moves.

In a steady, 2D flow, without heat sources,

$$\boxed{u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}} + \cancel{\alpha\frac{\partial^2 T}{\partial x^2}}$$

neglect, since $\frac{\partial^2 T}{\partial x^2} \ll \frac{\partial^2 T}{\partial y^2}$ in B.L.

Derivation : (1D) control volume



$$\Delta x\dot{q} + Q_{in} - Q_{out} = \text{energy gain}$$

$$Q_{in} = -k\frac{dT}{dx}\Big|_x + u\cdot\rho C_p T_{in}\Big|_x$$

$$Q_{out} = -k\frac{dT}{dx}\Big|_{x+\Delta x} + u\cdot\rho C_p T_{out}\Big|_{x+\Delta x}$$

$$\text{energy gain} = \rho C_p\frac{dT}{dt}\cdot\Delta x$$

$$\Rightarrow \dot{q}\Delta x - k\frac{\partial T}{\partial x}\Big|_x + u\cdot\rho C_p T_{in}\Big|_x + k\frac{\partial T}{\partial x}\Big|_{x+\Delta x} - u\rho C_p T_{out}\Big|_{x+\Delta x} = \rho C_p\frac{\partial T}{\partial t}\cdot\Delta x$$

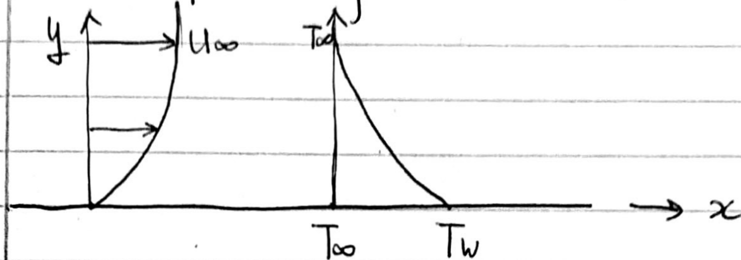
$$\dot{q} + k\frac{\left(\frac{\partial T}{\partial x}\Big|_{x+\Delta x} - \frac{\partial T}{\partial x}\Big|_x\right)}{\Delta x} + u\rho C_p\cdot\frac{\left(T\big|_x - T\big|_{x+\Delta x}\right)}{\Delta x} = \rho C_p\frac{\partial T}{\partial t}$$

$$\dot{q} + k\frac{\partial^2 T}{\partial x^2} - u\rho C_p\frac{\partial T}{\partial x} = \rho C_p\frac{\partial T}{\partial t}$$

$$\rho C_p\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x}\right) = k\frac{\partial^2 T}{\partial x^2} + \dot{q}$$

We found that 2D steady energy equation looks very similar to 2D momentum equation (with no pressure variation)!

Consider B.L. in a fluid with bulk U_∞ and T_∞ flowing over a flat surface at T_w, without a pressure gradient.



kinetic viscosity

momentem :
$$\boxed{u\frac{\partial}{\partial x}\left(\frac{u}{U_\infty}\right) + v\frac{\partial}{\partial y}\left(\frac{u}{U_\infty}\right) = \nu\frac{\partial^2}{\partial y^2}\left(\frac{u}{U_\infty}\right)} \quad \cdots \text{①}$$

B.C.
$$\begin{cases} y=0, & \frac{u}{U_\infty}=0 \\ y=\infty, & \frac{u}{U_\infty}=1 \\ y=\infty, & \frac{\partial}{\partial y}\left(\frac{u}{U_\infty}\right)=0 \end{cases}$$

$\frac{u}{U_\infty}$: dimensionless velocity

energy : define dimensionless temperature $\Theta = \frac{T - T_w}{T_\infty - T_w}$

$$\boxed{u\frac{\partial \Theta}{\partial x} + v\frac{\partial \Theta}{\partial y} = \alpha\frac{\partial^2 \Theta}{\partial y^2}} \quad \cdots \text{②}$$

thermal diffusivity

B.C.
$$\begin{cases} y=0, & \Theta=0 \\ y=\infty, & \Theta=1 \\ y=\infty, & \frac{\partial \Theta}{\partial y}=0 \end{cases}$$

equation ① and ② are identical with the same B.C.
only difference is ν and α!

The solution to ① is $\frac{u}{u_\infty} = f'(\eta)$, $\left\{\begin{array}{l}\text{where } \eta = \sqrt{\frac{u_\infty}{\nu x}}\, y \\ \text{and } f(\eta) \text{ (Blasius problem)}\end{array}\right.$

If $\alpha = \nu$, then Θ is the same with $\frac{u}{u_\infty}$.

We can immediately calculate h:

$$h = \frac{k}{T_\infty - T_w} \left.\frac{\partial (T - T_w)}{\partial y}\right|_{y=0} = k \left.\frac{\partial \Theta}{\partial y}\right|_{y=0} = k \left.\overbrace{\frac{\partial^2 f}{\partial \eta^2}}^{=0.332} \cdot \frac{\partial \eta}{\partial y}\right|_{\eta=0}$$

$$\frac{\partial \eta}{\partial y} \rightarrow \frac{\sqrt{Re_x}}{x}$$

$$\boxed{h = 0.332 \cdot k \cdot \frac{\sqrt{Re_x}}{x}}$$

or,

$$\boxed{\frac{hx}{k} = Nu_x = 0.332\sqrt{Re_x}}$$